۱-۷: عملگرهای دیفرانسیلی Differential Operators

مشتق تانسور $\tilde{T}$ به صورت زیر می‌شود.

$$\tilde{T} = \tilde{T}(t)$$

$$\lim_{\Delta t \to 0} \frac{\Delta \tilde{T}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\tilde{T}(t + \Delta t) - \tilde{T}(t)}{\Delta t} = \frac{d\tilde{T}}{dt}$$

$$\frac{d}{dt}\left(\tilde{T} \pm \tilde{S}\right) = \frac{d\tilde{T}}{dt} \pm \frac{d\tilde{S}}{dt}$$

$$\frac{d}{dt}\left(\alpha \tilde{T}\right) = \alpha \frac{d\tilde{T}}{dt} \;,\; \alpha = const. \quad \& \quad \frac{d}{dt}\left(\alpha(t) \tilde{T}\right) = \frac{d\alpha}{dt}\tilde{T} + \alpha \frac{d\tilde{T}}{dt}$$

$$\frac{d}{dt}\left(\tilde{T}\,\vec{a}\right)=\frac{d\tilde{T}}{dt}\vec{a}+\tilde{T}\,\frac{d\vec{a}}{dt}$$

$$\frac{d}{dt}\left(\tilde{T}\tilde{S}\right)=\frac{d\tilde{T}}{dt}\tilde{S}+\tilde{T}\,\frac{d\tilde{S}}{dt}$$

$$\frac{d}{dt}\left(\tilde{T}^{T}\right)=\left(\frac{d\tilde{T}}{dt}\right)^{T}$$

مثال: نشان دهید که مؤلفه‌های مشتق تانسور، برابر با مشتق مؤلفه‌های تانسور است.

$$\left(\frac{d\tilde{T}}{dt}\right)_{ij}\equiv\frac{dT_{ij}}{dt}$$

$$T_{ij}=\hat{e}_i\cdot\tilde{T}\hat{e}_j$$

$$\frac{dT_{ij}}{dt}=\frac{d}{dt}\left(\hat{e}_i\cdot\tilde{T}\hat{e}_j\right)=\hat{e}_i\cdot\frac{d}{dt}\left(\tilde{T}\hat{e}_j\right)=\hat{e}_i\cdot\left(\frac{d\tilde{T}}{dt}\right)\hat{e}_j=\left(\frac{d\tilde{T}}{dt}\right)_{ij}$$

تمرین: نشان دهید که برای تانسور متعامد $\tilde{Q}(t)$ عبارت $\frac{d\tilde{Q}}{dt}\tilde{Q}^{T}$ یک تانسور پادمتقارن است.

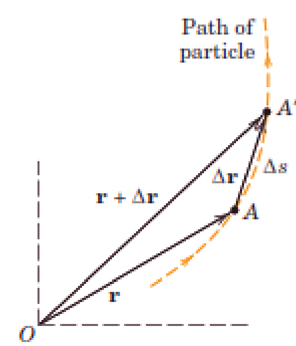
$$\text{position } \vec{r}=x_1\hat{e}_1+x_2\hat{e}_2+x_3\hat{e}_3=x_i\hat{e}_i$$

$$\vec{r}=r\hat{e}=x_i\hat{e}_i$$

$$r=\sqrt{x_ix_i}\ ,\ \hat{e}=\frac{\vec{r}}{r}=\frac{x_i}{r}\hat{e}_i$$

$$\frac{x_i}{r}=\alpha_i=\cos\left(\hat{e}_i,\vec{r}\right)\ \rightarrow\ x_i=\alpha_i r$$

$$\alpha_i\alpha_i=\frac{x_ix_i}{r^2}=1\ \ \&\ \ |\hat{e}|=\frac{|\vec{r}|}{r}=1\ \ \textit{unit vector}$$



$$d\vec{r}=dx_i\hat{e}_i$$

$$|d\vec{r}|^2=ds^2=d\vec{r}\cdot d\vec{r}=\left(dx_i\hat{e}_i\right)\cdot\left(dx_j\hat{e}_j\right)=\delta_{ij}dx_idx_j=dx_idx_i\ \Rightarrow\ ds=\sqrt{dx_idx_i}$$

۱-۷-۱: گرادیان Gradient

$$operator\ \vec{\nabla}=\frac{\partial}{\partial x_1}\hat{e}_1+\frac{\partial}{\partial x_2}\hat{e}_2+\frac{\partial}{\partial x_3}\hat{e}_3=\frac{\partial}{\partial x_i}\hat{e}_i$$

Scalar field $\phi(x_1,x_2,x_3)$

$$d\phi=\frac{\partial\phi}{\partial x_1}dx_1+\frac{\partial\phi}{\partial x_2}dx_2+\frac{\partial\phi}{\partial x_3}dx_3=\frac{\partial\phi}{\partial x_i}dx_i=\phi_{,i}dx_i$$

$$d\phi=\frac{\partial\phi}{\partial x_i}dx_i=\delta_{ij}\frac{\partial\phi}{\partial x_i}dx_j=\left(\frac{\partial\phi}{\partial x_i}\hat{e}_i\right)\cdot\left(dx_j\hat{e}_j\right)=\vec{\nabla}\phi\cdot d\vec{r}$$

$$\vec{\nabla}\phi=grad\,\phi=\frac{\partial\phi}{\partial x_i}\hat{e}_i=\phi_{,i}\hat{e}_i$$

$$\vec{\nabla}\phi=\frac{\partial\phi}{\partial x_i}\hat{e}_i\quad\&\quad d\phi=\vec{\nabla}\phi\cdot d\vec{r}=\frac{\partial\phi}{\partial x_i}dx_i$$

گرادیان میدان عددی، یک بردار می‌شود.

Vector field $\vec{V}(x_1,x_2,x_3)$

$$d\vec{V}=\frac{\partial\vec{V}}{\partial x_1}dx_1+\frac{\partial\vec{V}}{\partial x_2}dx_2+\frac{\partial\vec{V}}{\partial x_3}dx_3=\frac{\partial\vec{V}}{\partial x_i}dx_i=\vec{V}_{,i}dx_i$$

$$d\vec{V}=\frac{\partial\vec{V}}{\partial x_j}dx_j=\delta_{jm}\frac{\partial\vec{V}}{\partial x_j}dx_m=\delta_{jm}\frac{\partial(v_i\hat{e}_i)}{\partial x_j}dx_m=\frac{\partial v_i}{\partial x_j}dx_m\delta_{jm}\hat{e}_i$$

$$=\frac{\partial v_i}{\partial x_j}dx_m\left(\hat{e}_j\cdot\hat{e}_m\right)\hat{e}_i=\frac{\partial v_i}{\partial x_j}\left(\hat{e}_i\circ\hat{e}_j\right)dx_m\hat{e}_m=\left(\vec{\nabla}\circ\vec{V}\right)d\vec{r}=\left(\vec{\nabla}\vec{V}\right)d\vec{r}$$

$$\vec{\nabla}\vec{V}=grad\vec{V}=\frac{\partial\vec{V}}{\partial x_j}\hat{e}_j=\frac{\partial v_i}{\partial x_j}\hat{e}_i\circ\hat{e}_j=v_{i,j}\hat{e}_i\circ\hat{e}_j$$

$$(\vec{\nabla}\vec{V})_{ij}=\frac{\partial v_i}{\partial x_j}=v_{i,j}$$

$$\vec{\nabla}\vec{V}=\frac{\partial v_i}{\partial x_j}\hat{e}_i\circ\hat{e}_j\quad\&\quad d\vec{V}=(\vec{\nabla}\vec{V})d\vec{r}=\frac{\partial v_i}{\partial x_j}dx_j\hat{e}_i$$

گرادیان میدان برداری، یک تانسور می‌شود.

✍ گرادیان، مرتبه‌ی تانسور را یکی افزایش می‌دهد.

$$\vec{\nabla}\phi = \frac{\partial \phi}{\partial x_i}\hat{e}_i = \begin{pmatrix} \partial\phi/\partial x_1 \\ \partial\phi/\partial x_2 \\ \partial\phi/\partial x_3 \end{pmatrix}$$

$$\vec{\nabla}\vec{V} = \frac{\partial v_i}{\partial x_j}\hat{e}_i \circ \hat{e}_j = \begin{bmatrix} \partial v_1/\partial x_1 & \partial v_1/\partial x_2 & \partial v_1/\partial x_3 \\ \partial v_2/\partial x_1 & \partial v_2/\partial x_2 & \partial v_2/\partial x_3 \\ \partial v_3/\partial x_1 & \partial v_3/\partial x_2 & \partial v_3/\partial x_3 \end{bmatrix}$$

مثال: اگر $\vec{q}$ بردار شار حرارتی، θ میدان دما و k ضریب هدایت گرمایی باشد، شار حرارتی را در نقاط $A(1,0)$ و $B\left(1/\sqrt{2},1/\sqrt{2},\right)$ به‌دست آورید. نمودارهای هم‌دما (ایزوترم) را رسم نمایید.

$$\vec{q} = -k\vec{\nabla}\theta \;\; , \;\; \theta(x_1,x_2) = 2\left(x_1^2 + x_2^2\right)$$

حلّ:

$$grad\,\theta = \vec{\nabla}\theta = \frac{\partial\theta}{\partial x_i}\hat{e}_i = 4\left(x_1\hat{e}_1 + x_2\hat{e}_2\right)$$

$$\vec{q} = -k\vec{\nabla}\theta = -4k\left(x_1\hat{e}_1 + x_2\hat{e}_2\right)$$

$$\vec{q}_A = -4k\hat{e}_1$$

$$\vec{q}_B = -2\sqrt{2}k\left(\hat{e}_1 + \hat{e}_2\right)$$

$$x_1^2 + x_2^2 = \frac{\theta}{2}$$